

A PATTERN AVOIDANCE CRITERION FOR FREE INVERSION ARRANGEMENTS

WILLIAM SLOFSTRA

ABSTRACT. We show that the hyperplane arrangement of a coconvex set in a finite root system is free if and only if it is free in corank 4. As a consequence, we show that the inversion arrangement of a Weyl group element w is free if and only if w avoids a finite list of root system patterns. As a key part of the proof, we use a recent theorem of Abe and Yoshinaga to show that if the root system does not contain any factors of type C or F , then Peterson translation of coconvex sets preserves freeness. This also allows us to give a Kostant-Shapiro-Steinberg rule for the coexponents of a free inversion arrangement in any type.

1. INTRODUCTION

A central hyperplane arrangement \mathcal{A} in a complex vector space \mathbb{C}^l is said to be free if its module of derivations $\text{Der}(\mathcal{A})$ is free. If \mathcal{A} is free, then $\text{Der}(\mathcal{A})$ has a homogeneous basis of size l , and the degrees d_1, \dots, d_l of the generators are called the coexponents of \mathcal{A} . By a Theorem of Terao, the Poincare series $\sum t^i \dim H_i(\mathbb{C}^l \setminus \mathcal{A})$ of the complement of \mathcal{A} is then $\prod_i (1 + d_i t)$ [Ter81]. The best known free arrangements are the Coxeter arrangements, cut out by the root systems of finite Coxeter groups, for which the coexponents are equal to the exponents of the Coxeter group [Ter81].

It is natural to ask when a subarrangement of a Coxeter arrangement is free. For braid arrangements (i.e. Coxeter arrangements of type A), the subarrangements are graphic arrangements, and a theorem of Stanley states that a graphic arrangement is free if and only if the corresponding graph is chordal [ER94] [CDF⁺09]. For other types, the only known results focus on certain classes of subarrangements [ER94] [ST06] [ABC⁺14] [OPY08] [Slo13]. In particular if R is a finite crystallographic root system, and $S \subseteq R^+$ is a lower order ideal in dominance order, then the arrangement $\mathcal{A}(S)$ cut out by S is free by a theorem of Sommers and Tymoczko (all types except type E [ST06]) and Abe, Barakat, Cuntz, Hoge, and Terao (all types [ABC⁺14]). In this case, the coexponents of $\mathcal{A}(S)$ can be determined from the heights of the roots in S . If w is a rationally smooth element of the Weyl group $W(R)$, then the arrangement $\mathcal{A}(I(w))$ cut out by the inversion set $I(w)$ of w is also free, by a theorem of Oh, Postnikov, and Yoo (type A [OPY08]) and the author (all types [Slo13]). In this case, the coexponents of $\mathcal{A}(I(w))$ are equal to the exponents of w .

Let X be the flag variety of type R , and let $X(w)$ be the Schubert variety indexed by $w \in W(R)$. The maximal torus T of the underlying semisimple group acts on $X(w)$, and the T -fixed points are the elements x of $W(R)$ which are less than or equal to w in Bruhat order. The T -weights of the tangent space $T_w X(w)$ are the inversions $I(w) \subseteq R^+$ of w , while the T -weights in the tangent space $T_e X(w)$ at

the identity $e \in W(R)$ correspond to a lower order ideal $S(w) \subseteq R^+$. If $X(w)$ is smooth, then $I(w)$ and $S(w)$ have the same number of elements, and both $\mathcal{A}(I(w))$ and $\mathcal{A}(S(w))$ are free. A theorem of Akyildiz and Carrell [AC12] states that the coexponents of $\mathcal{A}(S(w))$ are equal to the exponents of w , so in the smooth case the coexponents of $\mathcal{A}(S(w))$ are equal to the coexponents of $\mathcal{A}(I(w))$. This leads to the question of whether $\mathcal{A}(I(w))$ and $\mathcal{A}(S(w))$ can be compared directly.

In this paper, we give an affirmative answer to this question using Peterson translation. If $x \leq y \leq w$, where $x = r_\alpha y$, then there is a T -invariant curve in $X(w)$ connecting x and y , and any T -submodule $M \subseteq T_y X(w)$ can be translated along this curve to a T -submodule of $T_x X(w)$ [CK03]. This geometric Peterson translation procedure defines a combinatorial Peterson translate on coconvex sets of R^+ . We say that a coconvex set S is Peterson-free if $\mathcal{A}(S)$ is free, and $\mathcal{A}(\tau(S))$ is free for all Peterson translates $\tau(S)$ of S . All lower order ideals are trivially Peterson-free, and an inversion set is Peterson-free if and only if it is free (the same is true for coconvex sets as long as R contains no factors of type C or F). What's more, the class of Peterson-free coconvex sets is closed under Peterson translation, and thus any Peterson-free coconvex set can be translated to a lower order ideal with the same coexponents. Consequently if $X(w)$ is smooth then we can directly compare $\mathcal{A}(I(w))$ and $\mathcal{A}(S(w))$ by repeatedly translating $I(w)$ from w to the identity. We extend these techniques further to show that the arrangement $\mathcal{A}(S)$ cut out by a coconvex set S is free if and only if the localizations $\mathcal{A}(S)_X$ are free for every flat of $\mathcal{A}(S)$ of corank ≤ 4 . This gives a root-system pattern avoidance criterion for the freeness of $\mathcal{A}(I(w))$. The proofs of these results are for the most part short, relying on a theorem of Abe and Yoshinaga [AY13] to reduce to low rank, where results can be checked by computer. The exception is type C_n , where we rely on a characterization of free subarrangements containing A_{n-1} due to Edelman and Reiner [ER94] to circumvent the fact that not all free coconvex sets in C_n are Peterson-free.

The rest of the paper is organized as follows. In Section 2, we state pattern avoidance criteria for the freeness of arrangements cut out by coconvex and inversion sets. In Section 3 we state and prove the main properties of combinatorial Peterson translation. In Section 4 we develop some additional tools, such as Peterson-freeness, to study Peterson translation in types C and F . In Section 5 we prove the pattern avoidance results, and explain some of the technical aspects of the computer checks used throughout the paper. Finally, in Section 6 we explain how the combinatorial results are related to the geometric Peterson translate.

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1.2. Notation. R will refer to a finite crystallographic root system in ambient Euclidean space V , with positive and negative roots R^+ and R^- respectively. We assume without loss of generality that R spans V . Given $S \subseteq R$ with the property that $S \cap -S = \emptyset$, we let $\mathcal{A}(S)$ denote the real arrangement in V^* cut out by S .

$L(\mathcal{A})$ will denote the intersection lattice of an arrangement \mathcal{A} . Given $X \in L(\mathcal{A})$, the localization \mathcal{A}_X is the subarrangement of \mathcal{A} consisting of those hyperplanes which contain X .

A real arrangement \mathcal{A} in a real vector space V^* has a complexification $\mathcal{A}_{\mathbb{C}}$ in $V_{\mathbb{C}}^* = V^* \otimes \mathbb{C}$. We say that a real arrangement \mathcal{A} is free if and only if $\mathcal{A}_{\mathbb{C}}$ is free. The intersection lattices $L(\mathcal{A})$ and $L(\mathcal{A}_{\mathbb{C}})$ are isomorphic, and complexification commutes with localization.

2. COCONVEX SETS, FREENESS, AND PATTERN AVOIDANCE

A subset S of R^+ is convex if $\alpha, \beta \in S$, $\alpha + \beta \in R^+$ implies that $\alpha + \beta \in S$, coconvex if $R^+ \setminus S$ is convex, and biconvex if S is convex and coconvex. The main result of this paper is a criterion for $\mathcal{A}(S)$ to be free when S is a coconvex set. Let X be a flat of an arrangement \mathcal{A} . Since X is the center of \mathcal{A}_X , the rank of \mathcal{A}_X is equal to the corank of X . If \mathcal{A} is free, then so is \mathcal{A}_X [OT92, Theorem 4.37]. For arrangements $\mathcal{A}(S)$ of coconvex sets S , we show that this result can be reversed while only checking flats X of low corank (and hence arrangements $\mathcal{A}(S)_X$ of low rank).

Theorem 2.1. *Let S be a coconvex subset of R^+ . Then $\mathcal{A}(S)$ is free if and only if $\mathcal{A}(S)_X$ is free for every flat $X \in L(\mathcal{A}(S))$ of corank ≤ 4 .*

Every arrangement of rank ≤ 2 is free. If $\mathcal{A}(S)$ has rank ≥ 4 , then $\mathcal{A}(S)$ will be free in Theorem 2.1 if and only if $\mathcal{A}(S)_X$ is free for every flat X of corank 4. We will show in the proof that if R contains no factors of types D , E , or F , then $\mathcal{A}(S)$ is free if and only if \mathcal{A}_X is free for every flat X of corank 3. Theorem 2.1 will be proved in Section 5.

If U is a subspace of V , then $R_U = R \cap U$ is a root system with positive and negative roots $R_U^+ = R^+ \cap U$ and $R_U^- = R^- \cap U$ respectively. Let R_0 be another root system in an ambient space V_0 , and fix $S_0 \subseteq R_0^+$. We say that a subset $S \subseteq R^+$ contains the pattern (S_0, R_0) if there is a subspace U of V such that $R_U \cong R_0$, and this isomorphism identifies $S_U = S \cap U \subseteq R_U^+$ with S_0 . If S does not contain (S_0, R_0) , then we say that S avoids (S_0, R_0) . If a coconvex (resp. convex, biconvex) set S contains a pattern (S_0, R_0) , then S_0 is also coconvex (resp. convex, biconvex). We say that a pattern (S_0, R_0) is coconvex (resp. convex, biconvex) if S_0 is coconvex (resp. convex, biconvex).

Given an arbitrary subset $S \subseteq R^+$ and a subspace U , let $X' = \bigcap_{\alpha \in U} \ker \alpha$, and let X be the smallest flat of $\mathcal{A}(S)$ containing X' (if U is spanned by elements of S then $X = X'$, whereas in general X is the sum of X' plus the center of $\mathcal{A}(S)$). The arrangement $\mathcal{A}(S_U)$ in U^* is linearly isomorphic to $\mathcal{A}(S)_X/X'$, a localization followed by a quotient, and consequently $\mathcal{A}(S_U)$ is free if and only if $\mathcal{A}(S)_X$ is free. Thus if S contains the pattern (S_0, R_0) and $\mathcal{A}(S)$ is free then $\mathcal{A}(S_0)$ is also free. Conversely if S contains (S_0, R_0) and $\mathcal{A}(S_0)$ is not free then $\mathcal{A}(S)$ cannot be free. Theorem 2.1 states that $\mathcal{A}(S)$ is free if and only if S avoids every non-free pattern (S_0, R_0) in root systems R_0 of rank ≤ 4 . To determine exactly which patterns must be avoided, we look at minimal patterns:

Definition 2.2. We say that a pattern (S_0, R_0) is a minimal non-free pattern if $\mathcal{A}(S_0)$ is non-free, and there is no proper subspace U_0 of the ambient space V_0 such that $\mathcal{A}((S_0)_{U_0})$ is a non-free arrangement in U_0^* .

Note that if (S_0, R_0) is minimal, then the vectors in S_0 span the ambient space V_0 . Theorem 2.1 can now be phrased in terms of pattern avoidance.

Corollary 2.3. Let R be a finite crystallographic root system, and let $S \subseteq R^+$ be a coconvex set. Then $\mathcal{A}(S)$ is free if and only if S avoids the minimal non-free coconvex patterns (S_0, R_0) , all of which are contained in the root systems $R_0 = A_3, B_3, C_3, D_4$, and F_4 .

The notion of root system pattern avoidance defined above is inspired by root system pattern avoidance for Weyl groups. Let $W(R)$ denote the Weyl group of R . The inversion set $I(w)$ of $w \in W(R)$ is the set $\{\alpha \in R^+ : w^{-1}\alpha \in R^-\}$. The inversion set $I(w)$ uniquely identifies w , and a subset $S \subseteq R^+$ is biconvex if and only if $S = I(w)$ for some element $w \in W(R)$. An element $w \in W(R)$ is said to contain (resp. avoid) the pattern (w_0, R_0) if $I(w)$ contains (resp. avoids) the pattern $(I(w_0), R_0)$. Thus root system pattern avoidance for Weyl groups is equivalent to the definition of pattern avoidance for biconvex sets given above. Root system pattern avoidance has been used by Billey and Postnikov [BP05] to characterize rationally smooth elements in $W(R)$. In type A , root system pattern avoidance is roughly equivalent to the usual pattern avoidance for permutations.

We say that (w_0, R_0) is a minimal non-free pattern if $(I(w_0), R_0)$ is minimal non-free. Since Corollary 2.3 holds in particular for biconvex sets, we have:

Corollary 2.4. Let R be a finite crystallographic root system, and let $w \in W(R)$. Then $\mathcal{A}(I(w))$ is free if and only if w avoids the minimal non-free patterns (w_0, R_0) , all of which are contained in the root systems $R_0 = A_3, B_3, C_3, D_4$, and F_4 .

The number of minimal patterns in Corollaries 2.3 and 2.4 are given in Table 1. The minimal non-free patterns (w_0, R_0) are explicitly listed in Table 2, where we use the same Dynkin diagram labelling as in [BP05] and [Slo13], with s_i referring to the i th simple reflection. To save space in Table 2, we use the notation $[a, b, \dots]$ to refer to a list of optional terms, so for example $[a, b]c$ would refer to the three terms c , ac , and bc . Tables 1 and 2 were constructed using an exhaustive computer search, which we describe further in Section 5.

Given $\alpha \in R$, let $\check{\alpha}$ denote the coroot $2\alpha/(\alpha, \alpha)$ in the dual root system \check{R} . Given $S \subseteq R^+$, let $\check{S} = \{\check{\alpha} : \alpha \in S\} \subseteq \check{R}^+$. Then S is a biconvex set in R if and only if \check{S} is a biconvex set in \check{R} , and $\mathcal{A}(S) = \mathcal{A}(\check{S})$. Similarly if σ is a diagram automorphism of R , then σ sends biconvex sets to biconvex sets, and $\mathcal{A}(\sigma(S)) \cong \mathcal{A}(S)$. These two features can be seen in Table 2. For example, B_3 and C_3 are dual, so $W(B_3) \cong W(C_3)$ have the same minimal non-free biconvex patterns. The number of minimal non-free coconvex patterns is different, however. D_4 has a diagram automorphism σ of order three, and the elements $s_i s_j s_3 s_2 s_1 s_3 s_4 s_2 s_i s_j$ all lie in the same σ orbit. Finally, the root system F_4 is self-dual, with the corresponding automorphism on $W(F_4)$ given by a diagram automorphism σ' of the Coxeter group. The first pattern listed in

R_0	Weyl group elements w_0	Coconvex sets S_0
A_3	1	3
B_3	7	42
C_3	7	50
D_4	4	21
F_4	3	391

TABLE 1. The number of minimal non-free patterns (w_0, R_0) (resp. (S_0, R_0)).

R_0	Elements w_0
A_3	$s_2 s_1 s_3 s_2$
B_3/C_3	$[s_3] s_2 s_1 s_3 s_2 [s_3], s_2 s_1 s_3 s_2 s_1 s_3 [s_2], s_1 s_3 s_2 s_1 s_3 s_2$
D_4	$s_2 s_1 s_3 s_4 s_2, s_i s_j s_2 s_1 s_3 s_4 s_2 s_i s_j, i, j \in \{1, 3, 4\}, i < j$
F_4	$s_4 s_3 s_2 s_3 s_4 s_1 s_2 s_3 s_4 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1, s_2 s_1 s_4 s_3 s_4 s_2 s_3 s_1, s_3 s_4 s_1 s_2 s_1 s_3 s_2 s_4$

TABLE 2. List of elements $w_0 \in W(R_0)$ such that (w_0, R_0) is a minimal non-free pattern.

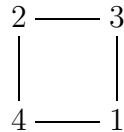
Table 2 for F_4 is invariant under σ' , while the last two lie in the same σ' -orbit. Also note that if S contains a pattern (S_0, R_0) , and σ is a diagram automorphism of S_0 , then S also contains $(\sigma(S_0), R_0)$, so technically there is some redundancy in listing two minimal non-free patterns in the same σ -orbit.

Example 2.5. Let e_1, \dots, e_{n+1} denote the standard basis of \mathbb{R}^{n+1} . The root system A_n is usually presented as $R = \{e_j - e_i : i \neq j\}$ in ambient space $V = \text{span}\{e_{i+1} - e_i : i = 1, \dots, n\}$. In this presentation, we take $R^+ = \{e_j - e_i : 1 \leq i < j \leq n+1\}$. Any subset $S \subseteq R^+$ corresponds to a graph $G(S)$ with vertex set $\{1, \dots, n+1\}$ and edge set $\{ij : e_j - e_i \in S\}$. As mentioned in the introduction, the arrangement $\mathcal{A}(S)$ is free if and only if $G(S)$ is chordal, meaning that every cycle of length ≥ 4 has a chord.

Let s_i be the simple reflection corresponding to simple root $\alpha_i = e_{i+1} - e_i$, and let $w = s_2 s_1 s_3 s_2$ in $W(A_3)$. Then

$$I(w) = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} = \{e_3 - e_2, e_3 - e_1, e_4 - e_2, e_4 - e_1\}.$$

The associated graph $G(I(w))$ is



Thus $G(I(w))$ is not chordal, and w is the unique element of $W(A_3)$ for which $\mathcal{A}(I(w))$ is not free.

3. COMBINATORIAL PETERSON TRANSLATION

In this section we introduce a translation procedure on coconvex subsets $S \subseteq R^+$ which is analogous to Peterson translation on the tangent space of a Schubert variety.

This procedure is essential to the proof of Theorem 2.1, and can be used to calculate coexponents of free inversion arrangements. However, to see that this procedure is correct in types C and F we rely on the more sophisticated analysis presented in the next section.

Let \preceq denote the dominance ordering on R^+ , so $\alpha \preceq \beta$ if and only if $\beta - \alpha$ is a non-negative linear combination of simple roots. A subset $S \subseteq R^+$ is a lower order ideal of R^+ if $\beta \in S$, $\alpha \preceq \beta$ implies that $\alpha \in S$. All lower order ideals are coconvex. If R^+ has simple roots $\{\alpha_i\}$, and $\alpha = \sum n_i \alpha_i \in R^+$, then the height of α is $\sum n_i$. Given a lower order ideal S , let h_i , $i \geq 1$, denote the number of elements of height i , and set $h_0 = l$, where l is the rank of R . In a lower order ideal, the number $h_i - h_{i+1}$ is always non-negative, so we can define the exponent set $\text{Exp}(S)$ of S to be a multi-set of non-negative integers in which i appears with multiplicity $h_i - h_{i+1}$. If $S = R^+$ then $\text{Exp}(S)$ is the set of exponents of W by a theorem of Kostant [Kos59], and this is sometimes called the Kostant-Macdonald-Shapiro-Steinberg rule for the exponents of W . As mentioned in the introduction, we will need:

Theorem 3.1 ([ST06], [ABC⁺14]). *If S is a lower order ideal in R^+ , then $\mathcal{A}(S)$ is free with coexponents $\text{Exp}(S)$.*

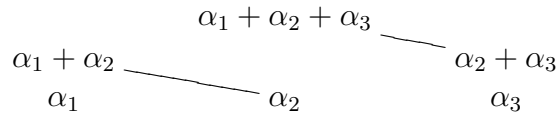
Given $\alpha \in R^+$, an α -string is a subset of R^+ of the form $\{\beta, \beta + \alpha, \beta + 2\alpha, \dots, \beta + k\alpha\}$, where $\beta - \alpha \notin R^+$ and $\beta + (k+1)\alpha \notin R^+$. The set of α -strings partitions R^+ . The Peterson translate of a subset $S \subseteq R^+$ compresses each α -string:

Definition 3.2. *Given $S \subseteq R^+$, $\alpha \in R^+$, we define the Peterson translate $\tau(S, \alpha)$ of S by α as follows:*

- *If S is a subset of an α -string $\{\beta, \beta + \alpha, \dots, \beta + k\alpha\}$ in R^+ , so $S = \{\beta + i_1\alpha, \dots, \beta + i_r\alpha\}$, then $\tau(S, \alpha) = \{\beta, \beta + \alpha, \dots, \beta + (r-1)\alpha\}$.*
- *For a general subset S of R^+ , let $S = \bigcup S_i$ be the partition of S induced by the partition of R^+ into α -strings. Then $\tau(S, \alpha) = \bigcup \tau(S_i, \alpha)$.*

In Section 6 we will show that this definition is equivalent to a geometric formula for Peterson translation given by Carrell and Kuttler [CK03].

Example 3.3. *In type A , all α -strings have size 1 or 2. Using the notation of Example 2.5, the α_1 -strings can be arranged as*



where roots in the same string are joined by a line. As examples of Peterson translation, we have

$$\begin{aligned}
 \tau(\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}, \alpha_1) &= \{\alpha_1, \alpha_2, \alpha_2 + \alpha_3\}, \\
 \tau(\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}, \alpha_1) &= \{\alpha_1, \alpha_2, \alpha_2 + \alpha_3\}, \\
 \tau(\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3\}, \alpha_1) &= \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3\}, \text{ and} \\
 \tau(\{\alpha_1, \alpha_2, \alpha_3\}, \alpha_1) &= \{\alpha_1, \alpha_2, \alpha_3\}.
 \end{aligned}$$

Proposition 3.4. *Let S be a coconvex set in R^+ . Then:*

- (a) The Peterson translation $\tau(S, \alpha)$ is coconvex for every $\alpha \in R^+$.
- (b) Suppose $\alpha \in R^+$, where α is either a long root, or is not contained in a factor of type C or F . If $\mathcal{A}(S)$ is free then $\mathcal{A}(\tau(S, \alpha))$ is free with the same coexponents as $\mathcal{A}(S)$.
- (c) If S is not a lower order ideal, then there is $\alpha \in S$ such that $\tau(S, \alpha)$ is not equal to S .

The proof of Proposition 3.4 is given later in this section. If S is coconvex and $\alpha \notin S$ then $\tau(S, \alpha) = S$, so we focus on translations by roots $\alpha \in S$.

Example 3.5. If S is not coconvex then part (b) of Proposition 3.4 does not hold, even if $\alpha \in S$. Let $S_0 = \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\} \subseteq A_3$. Then

$$S_1 := \tau(S_0, \alpha_2) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\},$$

so

$$G(S_0) = \begin{array}{ccc} 1 & \text{---} & 4 \\ & \searrow & \uparrow \\ 2 & \text{---} & 3 \end{array} \quad \text{while} \quad G(S_1) = \begin{array}{ccc} 1 & \text{---} & 4 \\ \uparrow & & \uparrow \\ 2 & \text{---} & 3 \end{array}.$$

Thus $\mathcal{A}(S_0)$ is free, but $\mathcal{A}(S_1)$ is not free.

To prove Proposition 3.4, we need some facts about multi-restriction of arrangements. Given an arrangement \mathcal{A} in V^* , the restriction \mathcal{A}^H to a hyperplane $H \in \mathcal{A}$ is the arrangement $\{K \cap H : K \in \mathcal{A} \setminus \mathcal{A}_H\}$ in H . A multi-arrangement is an arrangement \mathcal{A} , along with a multiplicity function $m : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 1}$ which assigns a multiplicity $m(H)$ to every hyperplane $H \in \mathcal{A}$. Equivalently, (\mathcal{A}, m) can be regarded as the (non-reduced) scheme cut out by the polynomial

$$Q = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)},$$

where α_H is a defining form for $H \in \mathcal{A}$. Given an arrangement \mathcal{A} , we let $\tilde{\mathcal{A}}$ denote the multi-arrangement where every hyperplane in \mathcal{A} has multiplicity one. The Zeigler multi-restriction $\tilde{\mathcal{A}}^H$ is the multi-arrangement with underlying arrangement \mathcal{A}^H , and multiplicity function $m(K) = |\{K' \in \mathcal{A} : K' \cap H = K\}|$.

Let S^*V denote the space of polynomials on V^* . The module of derivations $\text{Der}(\mathcal{A}, m)$ of the multi-arrangement is the space of derivations of S^*V which preserve the defining ideal generated by the polynomial Q . For an ordinary arrangement, the module of derivations $\text{Der}(\mathcal{A})$ is just $\text{Der}(\tilde{\mathcal{A}})$. A multi-arrangement (\mathcal{A}, m) is said to be free if $\text{Der}(\mathcal{A}, m)$ is a free S^*V -module. Zeigler showed that if \mathcal{A} is free, then $\tilde{\mathcal{A}}^H$ is free for any $H \in \mathcal{A}$ [Zie89]. Abe and Yoshinaga prove the following converse to Zeigler's theorem:

Theorem 3.6 ([AY13], Theorem 4.1). *Let \mathcal{A} be an arrangement such that $\tilde{\mathcal{A}}^H$ is free for some $H \in \mathcal{A}$. Then \mathcal{A} is free if and only if \mathcal{A}_X is free for every flat $X \subseteq H$ of corank 3.*

If v_i is a basis for V , then the Euler derivation $\sum v_i \frac{\partial}{\partial v_i}$ always lies in $\text{Der}(\mathcal{A})$, and hence if \mathcal{A} is free, then 1 appears as a coexponent with positive multiplicity. If we write the coexponents of \mathcal{A} as $1, m_2, \dots, m_l$, then Ziegler showed that the coexponents of $\tilde{\mathcal{A}}^H$ are m_2, \dots, m_l . This leads to:

Corollary 3.7 ([AY13], Corollary 4.3). *Suppose \mathcal{A}_1 and \mathcal{A}_2 are hyperplane arrangements, and let H_1 and H_2 be hyperplanes in \mathcal{A}_1 and \mathcal{A}_2 respectively. Suppose $\tilde{\mathcal{A}}_1^{H_1} \cong \tilde{\mathcal{A}}_2^{H_2}$, and that \mathcal{A}_1 is free. Then the following are equivalent:*

- (a) \mathcal{A}_2 is free, with the same coexponents as \mathcal{A}_1 .
- (b) $(\mathcal{A}_2)_X$ is free for every flat $X \subseteq H_2$ of corank 3.

The key lemma we need to prove Proposition 3.4 is the following:

Lemma 3.8. *Let $S_0 \subseteq R^+$, and $H = \ker \alpha$ for some $\alpha \in S$. Let $S_1 = \tau(S_0, \alpha)$. Then*

$$\widetilde{\mathcal{A}(S_0)}^H \cong \widetilde{\mathcal{A}(S_1)}^H.$$

Proof. The elements of S_1 are α -translates of the elements of S_0 , occurring with the correct multiplicities. \square

Proof of Proposition 3.4. To start, we observe that parts (a) and (b) of Proposition 3.4 hold for all root systems of rank ≤ 3 . Indeed, it is enough to check $R = A_3, B_3, C_3$, and G_2 , which we can do on a computer, using the methodology described in Section 5 to check freeness.

Now consider a general root system R , and let S be a coconvex subset of R^+ . Recall that if U is a subspace of V , then $S_U = S \cap U$. The key idea of the proof is that Peterson translation is local, in the sense that if $\alpha \in U$ then $\tau(S, \alpha)_U = \tau(S_U, \alpha)$. Suppose $\beta, \gamma \in R^+ \setminus \tau(S, \alpha)$ such that $\beta + \gamma \in R^+$. Let U be the subspace spanned by α, β , and γ . Then S_U is a coconvex subset of R_U^+ , where R_U is a root system of rank ≤ 3 , so $\tau(S_U, \alpha) = \tau(S, \alpha)_U$ is coconvex. Hence $\beta + \gamma \notin \tau(S, \alpha)_U$. It follows that $\tau(S, \alpha)$ is coconvex.

Similarly suppose that $\mathcal{A}(S)$ is free, and let X be a flat of $\mathcal{A}(\tau(S, \alpha))$ of corank 3 contained in $H = \ker \alpha$. Let $U \subseteq V$ be the subspace of linear forms that vanish on X . Then U has dimension 3, and $\alpha \in U$. Furthermore, if α is long in R , then α is long in R_U . Since $\mathcal{A}(S)$ is free, $\mathcal{A}(S_U)$ is also free, and thus $\mathcal{A}(\tau(S_U, \alpha)) = \mathcal{A}(\tau(S, \alpha)_U) = \mathcal{A}(\tau(S, \alpha))_X$ is free. Applying Lemma 3.8 and Corollary 3.7, part (b) we get that $\mathcal{A}(\tau(S, \alpha))$ is free with the same coexponents as $\mathcal{A}(S)$.

Finally, suppose that S is a coconvex set but is not a lower order ideal. This means that there is $\beta \in S$ and $\gamma \in R^+$ such that $\beta - \gamma \notin S$. Since S is coconvex, γ must be in S , and $\tau(S, \gamma)$ will include some element $\beta - c\gamma \in R^+$, $c > 0$, which is not in S . \square

If $\tau(S, \alpha) \neq S$, then the sum of the heights of roots in $\tau(S, \alpha)$ is strictly less than the sum of the heights of roots in S . Given a coconvex set S_0 which is not an order ideal, Proposition 3.4 implies that we can translate by $\alpha_0 \in S_0$ to get a different coconvex set $S_1 = \tau(S_0, \alpha_0)$. Repeating this procedure, we must eventually arrive at a lower order ideal, leading to the following corollary of Theorem 3.1 and Proposition 3.4.

Corollary 3.9. *If S is a coconvex set, then there is a sequence $S = S_0, \dots, S_r$ of distinct coconvex sets, such that $S_{i+1} = \tau(S_i, \alpha_i)$ for some $\alpha_i \in S_i$, and S_r is a lower order ideal. If $\mathcal{A}(S)$ is free, and R has no factors of type C or F , then $\mathcal{A}(S)$ has coexponents $\text{Exp}(S_r)$.*

In the next section we will show that we can remove the type restriction in Corollary 3.9 if S is an inversion set.

Example 3.10. *Using the notation of Example 2.5, let $w = s_1 s_2 s_3 s_2 s_1$ in $W(A_3)$. Then*

$$I(w) = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} = R^+ \setminus \{\alpha_2\}.$$

Since $w \neq s_2 s_1 s_3 s_2$, $\mathcal{A}(I(w))$ is free (alternatively, the associated graph is the complete graph minus an edge). We have

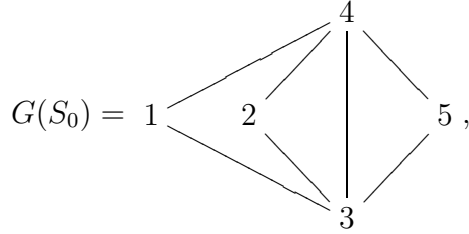
$$\begin{aligned} \tau(I(w), \alpha_1) &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} =: S_1, \text{ and} \\ \tau(S_1, \alpha_3) &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\} =: S_2. \end{aligned}$$

The set S_2 is an order ideal, and $\text{Exp}(S_2) = \{1, 2, 2\}$, so $\mathcal{A}(I(w))$ has coexponents $\{1, 2, 2\}$.

Example 3.11. *Although Peterson translation preserves freeness in type A , it is not a matroid invariant. Let*

$$S_0 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3\} \subseteq A_4^+.$$

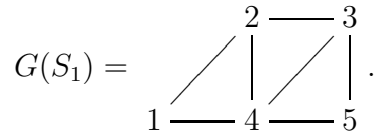
The set S_0 is coconvex, and the graph of S_0 is



so $\mathcal{A}(S_0)$ is free. Now

$$S_1 := \tau(S_0, \alpha_2) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3\},$$

and



The vector matroids of S_0 and S_1 are the cycle matroids of $G(S_0)$ and $G(S_1)$ respectively. The graph $G(S_1)$ has a simple 5-cycle, while $G(S_0)$ does not, so the cycle matroids cannot be isomorphic. However, $\mathcal{A}(S_1)$ is still free.

4. PETERSON-FREENESS AND THE PETERSON TRANSLATION GRAPH

In this section we further analyze Peterson translation in types C and F . We start by recalling the definition of Peterson-freeness from the introduction.

Definition 4.1. *We say that a coconvex set S is Peterson-free if $\mathcal{A}(S)$ is free, and $\mathcal{A}(\tau(S, \alpha))$ is free for any $\alpha \in S$.*

As in the previous section, if S is Peterson-free then $\mathcal{A}(\tau(S, \alpha))$ automatically has the same exponents as $\mathcal{A}(S)$ for all $\alpha \in R^+$.

Proposition 4.2. *Let S be a coconvex set such that $\mathcal{A}(S)$ is free. Then the following are equivalent:*

- (a) S is Peterson-free.
- (b) S_U is Peterson-free in R_U for any subspace $U \subseteq V$.
- (c) S_U is Peterson-free for every subspace $U \subseteq V$ of dimension 3.

Proof. To show that (a) implies (b), suppose that S_U is not Peterson-free for some subspace U . Then there is $\alpha \in S_U$ such that $\mathcal{A}(\tau(S_U, \alpha)) = \mathcal{A}(\tau(S, \alpha)_U)$ is not free. It follows that $\mathcal{A}(\tau(S, \alpha))$ is not free, and hence S is not Peterson-free.

It is clear that (b) implies (c). To finish the proof, we show that (c) implies (a). Suppose S is not Peterson-free. Since $\mathcal{A}(S)$ is free, there is $\alpha \in S$ such that $\mathcal{A}(\tau(S, \alpha))$ is not free. By Lemma 3.8 and Corollary 3.7, there must be a flat X of corank 3 contained in $\ker \alpha$ such that $\mathcal{A}(\tau(S, \alpha))_X$ is not free. If U is the space of linear forms vanishing on X , then $\mathcal{A}(S_U)$ is free and $\mathcal{A}(\tau(S_U, \alpha)) = \mathcal{A}(\tau(S, \alpha))_X$ is not. Thus S_U is not Peterson-free. \square

By Proposition 3.4, Peterson-freeness is equivalent to freeness if R has no factors of type C or F . Thus in part (c) of Proposition 4.2 it suffices to check subspaces U such that $R_U \cong C_3$. The positive roots of C_3 , arranged in order of height, are

$$\begin{array}{ccc} \alpha_1 + 2\alpha_2 + 2\alpha_3 & & \\ \alpha_1 + 2\alpha_2 + \alpha_3 & & \\ \alpha_1 + 2\alpha_2 & \alpha_1 + \alpha_2 + \alpha_3 & \\ \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

where α_1 is a long simple root, and α_2 and α_3 are short simple roots. Using a computer, we can easily check that the only coconvex sets S in C_3 for which $\mathcal{A}(S)$ is free but S is not Peterson-free are

$$\begin{aligned} S_1 &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2 + 2\alpha_3\}, \\ (4.1) \quad S_2 &= \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}, \text{ and} \\ S_3 &= \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}. \end{aligned}$$

If we add these three patterns to the 50 minimal non-free coconvex patterns in type C_3 , then we get a pattern avoidance characterization of Peterson-freeness, rather than freeness. Also, none of these sets are biconvex, so we immediately have the following:

Corollary 4.3. *If $S \subset R^+$ is an inversion set and $\mathcal{A}(S)$ is free, then S is Peterson-free.*

The most important property of Peterson-freeness is the following:

Proposition 4.4. *If $S \subset R^+$ is Peterson-free and $\alpha \in R^+$ then $\tau(S, \alpha)$ is also Peterson-free.*

Proof. Suppose $\mathcal{A}(S)$ is free, but $\tau(S, \alpha)$ is not Peterson-free for some $\alpha \in S$. We want to show that S is not Peterson-free. We can assume that $\mathcal{A}(\tau(S, \alpha))$ is free, since otherwise S is not Peterson-free by definition. With this assumption, Proposition 4.2 implies that there is a subspace $U \subseteq V$ of dimension 3 for which $\tau(S, \alpha)_U$ is not Peterson-free. Let $U' = U + \mathbb{R}\alpha$. Then $\tau(S, \alpha)_{U'} = \tau(S_{U'}, \alpha)$ is not Peterson-free by Proposition 4.2 again, and we will be done if we can show that $S_{U'}$ is not Peterson-free. Since U' has dimension ≤ 4 , it suffices to check Proposition 3.4 for root systems of rank ≤ 4 . Since freeness and Peterson-freeness are equivalent except in types C and F , we finish the proof by checking on a computer that the proposition holds in C_4 and F_4 . \square

It is helpful to think of Peterson translation and Peterson-freeness using a certain directed graph:

Definition 4.5. *Let R be a finite root system. The Peterson translation graph of R is the directed graph $\mathcal{G} \cong \mathcal{G}(R)$ with vertex set*

$$V(\mathcal{G}) = \{S \subseteq R^+ : S \text{ is coconvex}\},$$

and an edge $S_0 \rightarrow S_1$ if $S_1 = \tau(S_0, \alpha) \neq S_0$ for some $\alpha \in S_0$.

The graph \mathcal{G} is acyclic, and by Proposition 3.4, part (c) the terminal vertices of \mathcal{G} are the lower order ideals in R^+ . In terms of the Peterson translation graph, Proposition 4.4 is equivalent to:

Corollary 4.6. *A coconvex set $S \subset R^+$ is Peterson-free if and only if $\mathcal{A}(T)$ is free for every coconvex set T for which there is a directed path $S \rightsquigarrow T$ in $\mathcal{G}(R)$.*

In particular, if $\mathcal{A}(S)$ is free and S is an inversion arrangement then we can translate S to a lower order ideal T , and $\mathcal{A}(S)$ will have coexponents $\text{Exp}(T)$ regardless of type.

Remark 4.7. *It might be interesting to study the Peterson translation graph from a combinatorial perspective. For instance, it is possible to reach every lower order ideal by Peterson translation from an inversion set?*

5. PROOF OF PATTERN AVOIDANCE RESULTS

In this section, we prove the main pattern avoidance results from Section 2. In subsection 5.1 we also discuss our methodology for checking the many statements left up to computer verification in this and previous sections. We start by restricting our attention to those edges of $\mathcal{G}(R)$ which preserve freeness locally.

Definition 5.1. *Given a root system R , let $\mathcal{G}^{Fr}(R)$ be the subgraph of the Peterson translation graph $\mathcal{G}(R)$ with the same vertex set, but only those edges $S \rightarrow \tau(S, \alpha)$ for which there does not exist a subspace $\alpha \in U \subset V$ such that $\mathcal{A}(S_U)$ is free but $\mathcal{A}(\tau(S_U, \alpha))$ is not.*

By Lemma 3.8 and Corollary 3.7, to test whether $S \rightarrow \tau(S, \alpha)$ is an edge in $\mathcal{G}^{Fr}(R)$ it is enough to check subspaces $\alpha \in U \subset V$ of dimension 3. If R does not contain any factors of types C or F then $\mathcal{G}^{Fr}(R) = \mathcal{G}(R)$.

We now can define two properties of a root system R and fixed integer k :

- (L_k) If $S \subseteq R^+$ is coconvex, and $\mathcal{A}(S)_X$ is free for all flats X of corank $\leq k$, then $\mathcal{A}(S)$ is free.
- (T_k) If S is a terminal vertex of $\mathcal{G}^{Fr}(R)$, and $\mathcal{A}(S)_X$ is free for all flats X of corank $\leq k$, then $\mathcal{A}(S)$ is free.

The main result of this section is a criterion for (L_k) to hold for a class of root systems, assuming that (T_k) holds for all elements of the class.

Proposition 5.2. *Let \mathcal{C} be a class of finite root systems with the property that if $R \in \mathcal{C}$, and U is a subspace of the ambient space of R , then R_U is isomorphic to an element of \mathcal{C} . Suppose there is some $k \geq 3$ such that (T_k) holds for all irreducible root systems in \mathcal{C} , and (L_k) holds for all irreducible root systems in \mathcal{C} of rank $k + 1$. Then (L_k) holds for all $R \in \mathcal{C}$.*

Proof. First note that our hypothesis implies that (L_k) holds for all root systems in \mathcal{C} of rank $\leq k + 1$. Let $R \in \mathcal{C}$, and suppose $S \subseteq R^+$ is a coconvex set with the property that $\mathcal{A}(S)_X$ is free for all flats X of corank $\leq k$. This is equivalent to saying that $\mathcal{A}(S_U)$ is free for all subspaces $U \subseteq V$ spanned by at most k elements of S .

We show that $\mathcal{A}(S)$ is free by induction on the sum of the heights of the roots in S . If S is a terminal vertex in $\mathcal{G}^{Fr}(R)$, then $\mathcal{A}(S)$ is free by (T_k) . Otherwise, there is $\alpha \in S$ such that $S \rightarrow \tau(S, \alpha)$ is an edge in \mathcal{G}^{Fr} . Now suppose that U is a subspace of V spanned by at most k elements of $\tau(S, \alpha)$, and let $U' = U + \mathbb{R}\alpha$. Then $S_{U'}$ is a coconvex subset of $R_{U'}^+$, and $R_{U'}$ is a root system of rank $\leq k + 1$. If U'' is spanned by $\leq k$ elements of $S_{U'}$, then $\mathcal{A}((S_{U'})_{U''}) = \mathcal{A}(S_{U''})$ is free, and since (L_k) holds for $R_{U'}$ we conclude that $\mathcal{A}(S_{U'})$ is free. Since $\alpha \in U'$, $\mathcal{A}(\tau(S, \alpha)_{U'}) = \mathcal{A}(\tau(S_{U'}, \alpha))$, and $\mathcal{A}(\tau(S_{U'}, \alpha))$ is free by the definition of \mathcal{G}^{Fr} . Hence $\mathcal{A}(\tau(S, \alpha)_U)$ is free for all subspaces U spanned by at most k elements of $\tau(S, \alpha)$. Since the sum of the heights of the roots in $\tau(S, \alpha)$ is less than the sum of the heights of the roots in S , we conclude by induction that $\mathcal{A}(\tau(S, \alpha))$ is free. But since $k \geq 3$, we know that $\mathcal{A}(S_U)$ is free for all subspaces U of dimension ≤ 3 , and hence $\mathcal{A}(S)$ is free by Lemma 3.8 and Corollary 3.7, part (b). \square

Proof of Theorem 2.1 and Corollary 2.3. Clearly we only need to prove the results for irreducible root systems. For F_4 and G_2 there is nothing to be done, since these root systems have rank ≤ 4 . The root subsystems of an irreducible root system R are well-known; see for instance [Dyn57, Tables 9 and 10], as well as [DL11] for a modern account. The root systems R_U , for U a subspace of V , can be easily determined from these results. For instance, the restriction R_U of $R = B_n$ to a subspace is always a

direct sum of root systems of type A and B . Thus if \mathcal{C} is the class of root systems R whose irreducible factors are of types A or B , then the restriction R_U of a rootsystem $R \in \mathcal{C}$ to a subspace U is also in \mathcal{C} . By Proposition 3.4, $\mathcal{G}^{Fr}(R) = \mathcal{G}(R)$ for every $R \in \mathcal{C}$, and the terminal elements of $\mathcal{G}^{Fr}(R)$ are simply the lower order ideals in R , so the condition (T_k) holds for every $R \in \mathcal{C}$ (irregardless of k). We use a computer to check that (L_3) holds for A_4 and B_4 , after which Proposition 5.2 implies that (L_3) holds for all root systems in \mathcal{C} .

If \mathcal{C} is instead the class of simply-laced root systems (those whose irreducible factors are of types A , D , and E), we can make the exact same argument, except that (L_3) does not hold for D_4 . However, we can verify by computer that (L_4) holds for D_5 . Since we already know that (L_3) holds for every A_n , the condition (L_4) holds for A_5 , and Proposition 5.2 implies that (L_4) holds for all simply-laced root systems.

This leaves type C_n , which is the only classical type for which (T_k) does not hold trivially. Let \mathcal{C} be the class of root systems whose irreducible factors are of types A or C . This class is also closed under restriction to a subspace. We can verify on a computer that (L_3) holds for C_4 . To finish the argument, we show that (T_3) holds for all C_n , $n \geq 4$ (from which it follows immediately that (T_3) holds for all elements of \mathcal{C}).

We use the usual presentation of the root system $R = C_n$, with

$$C_n^+ = \{e_j - e_i : 1 \leq i < j \leq n\} \cup \{e_i + e_j : 1 \leq i \leq j \leq n\}.$$

With this choice of positive roots, the simple roots are $\alpha_1 = 2e_1$ and $\alpha_i = e_i - e_{i-1}$, $i = 2, \dots, n$. The root system A_{n-1} is canonically contained in C_n as the span of the simple roots $\alpha_2, \dots, \alpha_n$, or equivalently as the set of vectors of the form $e_j - e_i$, $i \neq j$. If $k \leq n$, we regard C_k and A_{k-1} as natural subsystems of C_n and A_{n-1} . We need three lemmas:

Lemma 5.3. *A root $\alpha \in C_n^+$ belongs to the subsystem A_{n-1}^+ if and only if there is a long root $\beta \in C_n^+$ such that if $U_0 = \text{span}\{\alpha, \beta\}$, then $R_{U_0} \cong C_2$ and α and β are the short and long simple roots of R_{U_0} respectively. Furthermore, if $\alpha \in A_{n-1}^+$ then the root β and subspace U_0 are unique.*

Proof. If α is short but not in A_{n-1} , then α must be of the form $e_i + e_j$ for $i \neq j$, and there is no positive long root β such that $\beta + \alpha \in R^+$. So if there is a subspace $U_0 \ni \alpha$ such that $R_{U_0} \cong C_2$ and α is the short simple root in R_{U_0} , then we must have $\alpha \in A_{n-1}$.

Conversely, if $\alpha = e_j - e_i$ for $i < j$ then there is a unique β such that $R_{U_0} \cong C_2$, namely $\beta = 2e_i$. \square

Lemma 5.4. *Suppose S is a terminal vertex in $\mathcal{G}^{Fr}(C_n)$, and α is a root in S such that $\tau(S, \alpha) \neq S$. Then α belongs to A_{n-1} and, in the notation from Lemma 5.3, one of the following conditions holds:*

- (a) $S_{U_0} = \{\alpha, \beta, \beta + 2\alpha\}$, or
- (b) $S_{U_0} = \{\alpha, \beta + \alpha\}$.

Proof. By the definition of \mathcal{G}^{Fr} , there must be a subspace U containing α such that $\mathcal{A}(S_U)$ is free and $\mathcal{A}(\tau(S_U, \alpha))$ is not. By Corollary 3.7 and Lemma 3.8, we can take

U to be of dimension 3. As discussed in Section 4, we must have $R_U \cong C_3$, and this isomorphism must send S_U to one of the coconvex sets $S_i \subset C_3$ from Equation (4.1). If S_U is sent to S_2 , then there is a long root β such that $\tau(S_U, \beta) \neq S_U$. Consequently $\tau(S, \beta) \neq S$, and by Proposition 3.4, S is not terminal. So S_U is sent to either S_1 or S_3 . The only roots α' in S_1 and S_3 for which $\tau(S_i, \alpha')$ is not free are α_2 , α_3 , and $\alpha_2 + \alpha_3$, so α must be sent to one of these roots in C_3 . Since all these roots belong to $A_2 \subseteq C_3$, it follows from Lemma 5.3 that there is $\beta' \in S_U$ such that β' and α form the simple roots of a subsystem $R_{U_0} \cong C_2$. Thus α lies in A_{n-1} , and the rest of the lemma follows from looking at the image of S_{U_0} inside of S_1 and S_3 . \square

Lemma 5.5. *Suppose S is a terminal vertex in $\mathcal{G}^{Fr}(C_n)$.*

- *If $e_k - e_i \in S$ for $1 \leq i < k$ then $e_j - e_i \in S$ for all $i < j < k$.*
- *If $e_l + e_k \in S$ for some $1 \leq l, k \leq n$, then $e_j - e_i \in S$ for all $1 \leq i < j \leq k$.*

Proof. We start by showing that if $e_l + e_k \in S$, then $e_k - e_i \in S$ for all $1 \leq i < k$. Indeed, if $e_i + e_l \notin S$ then $e_k - e_i = (e_l + e_k) - (e_i + e_l) \in S$ by coconvexity. If $e_i + e_l \in S$ then $\tau(S, e_i + e_l) = S$ by Lemma 5.4, and we get the same conclusion.

Now for the first part of the lemma, let $\alpha = e_k - e_j$, so $e_j - e_i = (e_k - e_i) - \alpha$. If $\alpha \notin S$ or $\tau(S, \alpha) = S$ then we are done, so suppose $\alpha \in S$ and $\tau(S, \alpha) \neq S$. Let $\beta = 2e_j$ and $U_0 = \text{span}\{\alpha, \beta\}$, so $R_{U_0} \cong C_2$ with simple roots α and β . If condition (a) holds in Lemma 5.4 then $\beta \in S_{U_0}$, while if condition (b) holds then $\beta + \alpha = e_k + e_j \in S_{U_0}$. In both cases $e_j - e_i \in S$ by first paragraph.

For the second part of the lemma, we know that $e_k - e_i \in S$ for $1 \leq i < k$, and hence $e_j - e_i \in S$ for all $1 \leq i < j \leq k$ by the first part of the lemma. \square

Suppose S is a terminal vertex of $\mathcal{G}^{Fr}(C_n)$. We can assume that S is not a lower order ideal, since otherwise $\mathcal{A}(S)$ is free by Theorem 3.1. We can also assume that $S \not\subseteq C_{n-1}$, or in other words that $S \setminus C_{n-1}^+$ is non-empty. We first consider the case that $S \setminus C_{n-1}^+$ is contained in A_{n-1}^+ . If $e_n - e_i, e_n - e_j \in S \setminus C_{n-1}^+$, where $1 \leq i < j < n$, then $e_j - e_i = (e_n - e_i) - (e_n - e_j) \in S$ by Lemma 5.5. Consequently

$$X = \bigcap_{\alpha \in S \cap C_{n-1}^+} \ker \alpha$$

is a modular coatom of $\mathcal{A}(S)$. Thus $\mathcal{A}(S)$ will be free if and only if $\mathcal{A}(S)_X = \mathcal{A}(S \cap C_{n-1}^+)$ is free.

Now suppose that $S \setminus C_{n-1}^+$ is not completely contained in A_{n-1}^+ . Then $e_l + e_n \in S$ for some $1 \leq l \leq n$, and hence $e_j - e_i \in S$ for all $1 \leq i < j \leq n$ by Lemma 5.5. The free arrangements $\mathcal{A}(S)$ where S contains A_{n-1}^+ have been characterized by Edelman and Reiner [ER94, Theorem 4.6]. In particular, they show that $\mathcal{A}(S)$ is free if $\mathcal{A}(S)_X$ is free for all flats X of corank ≤ 4 .¹ But since we already know that (L_3) holds for C_4 , it is enough to check that $\mathcal{A}(S)_X$ is free for all flats of corank ≤ 3 . Thus (T_3) holds for all C_n , finishing the proof. \square

¹In fact, their criterion applies even when S is not coconvex. To see that freeness follows from checking flats of corank ≤ 4 , use the first part of the proof of Theorem 4.6 combined with Theorem 4.1 and Lemma 4.5.

5.1. Determining freeness on a computer. Many of the proofs in this paper rely on computer verification. The positive roots of a given root system can be listed using common mathematical software, such as Maple (with John Stembridge's coxeter package) or the Sage Mathematics system. Given the positive roots, it is straightforward to list all coconvex sets or compute the Peterson translation of a coconvex set. Verifying the freeness of an arrangement $\mathcal{A}(S)$ is more complicated, so in this section we explain how this is done.

Given $H \in \mathcal{A}$, let $\mathcal{A} \setminus H$ denote the deletion of H from \mathcal{A} . An arrangement is said to be inductively free with exponents m_1, \dots, m_l if either \mathcal{A} is empty and $m_1 = \dots = m_l = 0$, or there is a hyperplane $H \in \mathcal{A}$ and index i such that $\mathcal{A} \setminus H$ is inductively free with exponents $m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_l$, and \mathcal{A}^H is inductively free with exponents $m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_l$. Inductive freeness can be checked on a computer in low rank using brute force, and in higher ranks using optimizations as in [Slo13].

Given a real arrangement \mathcal{A} , the Poincare polynomial of \mathcal{A} is

$$Q(\mathcal{A}, t) = \sum t^i \dim H^i(\mathcal{M}(\mathcal{A})),$$

where $\mathcal{M}(\mathcal{A})$ is the complement of the complexification of \mathcal{A} . If \mathcal{A} is free with coexponents m_1, \dots, m_l , then $Q(\mathcal{A}, t) = \prod_i (1 + m_i t)$ by a theorem of Terao [Ter81]. We say that $Q(\mathcal{A}, t)$ splits if it can be written in this form, for some non-negative integers m_1, \dots, m_l . The polynomial $Q(\mathcal{A}, t)$ can easily be computed using a restriction-deletion recurrence, and thus we can determine if $Q(\mathcal{A}, t)$ splits on a computer.

Definition 5.6. *We say that \mathcal{A} is verifiably free if \mathcal{A} is inductively free, and verifiably non-free if $Q(\mathcal{A}, t)$ does not split. We say that freeness of \mathcal{A} is verifiable if \mathcal{A} is verifiably free or non-free.*

It is not true that the freeness of $\mathcal{A}(S)$ is verifiable for every coconvex set S . However, freeness is always verifiable in A_3 , B_3 , and C_3 , and this suffices for the proof of Proposition 3.4.

To determine the minimal non-free coconvex patterns in a root system, we use the following procedure for each coconvex set S :

- Check whether or not $\mathcal{A}(S)$ is verifiably free. If $\mathcal{A}(S)$ is verifiably free, stop and return FREE.
- Otherwise, for each subspace U of the ambient space V with $3 \leq \dim U < \text{rank } \mathcal{A}(S)$, check whether or not the freeness of $\mathcal{A}(S_U)$ is verifiable.
 - If $\mathcal{A}(S_U)$ is verifiably non-free, stop and return NOT MINIMAL.
 - If freeness of $\mathcal{A}(S_U)$ is not verifiable, stop and return AMBIGUOUS.
 - Otherwise, continue.
- At the end of the loop, check whether or not $\mathcal{A}(S)$ is verifiably non-free. If so, return MINIMAL PATTERN.
- If, at this point, the freeness of $\mathcal{A}(S)$ is not verifiable, return AMBIGUOUS.

This procedure outputs AMBIGUOUS if $\mathcal{A}(S)$ is not verifiably free, and either there is a subspace U such that freeness of $\mathcal{A}(S_U)$ is not verifiable, or $\mathcal{A}(S_U)$ is verifiably free for all subspaces U and $\mathcal{A}(S)$ is not verifiably non-free. However, this procedure

does not output AMBIGUOUS for any of the root systems $A_3, B_3, C_3, D_4, F_4, A_4, B_4, C_4$, and D_5 . In particular, we can use this procedure to check that only the first five of these root systems have any minimal non-free coconvex patterns, from which we conclude that (L_3) holds in A_4-C_4 and (L_4) holds in D_5 .

We use a similar method to check freeness of coconvex sets in the proof of Proposition 4.4. Assuming that we can test for freeness, it is not hard to construct the Peterson translation graph, and from this graph determine all Peterson-free elements in types C_4 and F_4 .

The programs used to perform the computer verifications listed in this paper are available from the author's website. Implementations are provided in Maple and C++, with the C++ implementation running in a matter of minutes for D_5 .

6. GEOMETRIC INTERPRETATION OF THE PETERSON TRANSLATE

The combinatorial Peterson translation defined in the previous section comes from a geometric construction on the flag variety introduced by Peterson and further developed by Carrell and Kuttler [CK03]. Let G be the linear algebraic group associated to R , and fix a choice of Borel B and maximal torus $T \subseteq B$ compatible with the choice of positive roots R^+ . As in the introduction, let $X = G/B$ be the flag variety associated to R , and let $X(w)$ be the Schubert variety indexed by $w \in W(R)$. Let $x, y \in W(R)$ be two elements differing by a reflection, so $x = r_\alpha y$, where r_α is reflection through some root $\alpha \in R^+$. Suppose further that $x < y$ in Bruhat order, or equivalently that $\alpha \in I(y)$. There is a T -invariant curve $C \subseteq X$ with $C^T = \{x, y\}$, and any T -module $M \subseteq T_y X$ can be translated along C to a T -module $\tilde{\tau}(M, \alpha) \subseteq T_x X$, called the Peterson translate of M along C . This geometric Peterson translate has been used by Carrell and Kuttler to study smoothness for Schubert varieties, and we refer to [CK03] for more details. If $x, y \leq w$ in Bruhat order, then C is contained in $X(w)$, and hence if $M \subseteq T_y X(w)$ then $\tilde{\tau}(M, \alpha) \subseteq T_x X(w)$.

Given a T -module M , let ΩM denote the set of T -weights. Given $y \in W(R)$, we have

$$\Omega T_y X = y \Omega T_e X = y R^- = I(y) \cup \{\beta \in R^- : y^{-1} \beta \in R^-\}.$$

Given $S \subseteq \Omega T_y X$, there is a unique T -submodule $M \subseteq T_y X$ with $S = \Omega M$, and for any $\alpha \in I(y)$ we can define

$$\tilde{\tau}(S, \alpha) = \Omega \tilde{\tau}(M, \alpha) \subseteq \Omega T_x X,$$

where $x = r_\alpha y$. Carrell and Kuttler give an explicit formula for $\tilde{\tau}(S, \alpha)$, and this formula is the basis for the definition of Peterson translation in Section 3. The relation between the two notions of Peterson translation can be explicitly stated as follows:

Proposition 6.1. *With notation as above, we have*

$$(6.1) \quad -x^{-1} \tilde{\tau}(S, \alpha) = \tau(-y^{-1} S, -y^{-1} \alpha),$$

or in other words, the map $-y^{-1} : \Omega T_y X \rightarrow R^+$ transforms $\tilde{\tau}(\cdot, \alpha)$ to combinatorial Peterson translation by $-y^{-1} \alpha$.

Proof. Let S be a subset of $\Omega T_y X$, and let $\alpha \in I(y)$, $x = r_\alpha y$. Carrell and Kuttler give a formula for $\tilde{\tau}(S, \alpha)$ as follows: Let \mathfrak{g} be the Lie algebra of G , and let \mathfrak{g}_β be the root spaces of the roots $\beta \in R$. Let T^α be the kernel of the character $e^\alpha : T \rightarrow \mathbb{C}$ (the subtorus T^α is the stabilizer of a generic point in the curve C connecting y and x). Let M be the submodule of $T_y X$ with $\Omega M = S$. Note that since $T_e X$ is a $\mathfrak{g}_{-y^{-1}\alpha}$ -module, $T_y X$ is a $\mathfrak{g}_{-\alpha}$ -module. Carrell and Kuttler show that

$$(6.2) \quad \tilde{\tau}(S, \alpha) = r_\alpha \Omega M^{-\alpha},$$

where $M^{-\alpha}$ is the unique $\mathfrak{g}_{-\alpha}$ -submodule of $T_y X$ such that $M^{-\alpha}$ is isomorphic to M as a T^α -module. So

$$-x^{-1}\tilde{\tau}(S, \alpha) = -x^{-1}r_\alpha \Omega M^{-\alpha} = -y^{-1}\Omega M^{-\alpha}.$$

Let $\alpha' = -y^{-1}\alpha \in R^+$. Then $-y^{-1}$ gives a bijective correspondence between $\mathfrak{g}_{-\alpha}$ -submodules of $T_y X$ and $\mathfrak{g}_{-\alpha'}$ -submodules of $\mathfrak{n} = \mathfrak{g}/\mathfrak{b}^-$, where \mathfrak{b}^- is the negative Borel subalgebra of \mathfrak{g} . Furthermore, M and N are isomorphic as T^α -modules if and only if $-y^{-1}M$ and $-y^{-1}N$ are isomorphic as $T^{\alpha'}$ -modules. Finally, it is easy to see that if $M \subseteq \mathfrak{n}$, and N is an isomorphic $T^{\alpha'}$ -module which is a $\mathfrak{g}_{-\alpha'}$ -submodule of \mathfrak{n} , then $\Omega N = \tau(\Omega M, \alpha')$. Thus Carrell and Kuttler's formula in Equation (6.2) is equivalent to the formula in Equation (6.1). \square

It follows that we can apply the results of Sections 3 and 4 in this geometric setting.

Example 6.2. *As mentioned in the introduction, in [Slo13] it is shown that if w is rationally smooth, then $\mathcal{A}(I(w))$ is free with coexponents equal to the exponents of w . The exponents of w are integers m_1, \dots, m_l , where l is the rank of r , such that the Poincare series $P_w(q) = \sum_{x \leq w} q^{\ell(x)}$ for the Bruhat interval below w is equal to the product $\prod_i [m_i + 1]_q$, where $[k]_q$ is the q -integer $(1 + q + \dots + q^{k-1})$.*

The elements listed in Table 2 are all non-rationally smooth, and hence Corollary 2.4 gives another proof that $\mathcal{A}(I(w))$ is free when w is rationally smooth. Corollary 2.4 does not imply that the coexponents of $\mathcal{A}(I(w))$ are equal to the exponents of w . However, $S(w) := -\Omega T_e X(w)$ is a lower order ideal, and when $X(w)$ is smooth a theorem of Akyildiz and Carrell states that the exponents of w are equal to $\text{Exp } S(w)$. In this situation we can use Corollary 4.3 to show that the coexponents of $\mathcal{A}(I(w))$ are also equal to $\text{Exp } S(w)$. Choose a sequence of element $e = y_k < y_{k-1} < \dots < y_0 = w$, where $y_i = r_{\beta_i} y_{i-1}$ for some $\beta_i \in R^+$. As shown in [CK03], $\tilde{\tau}(\Omega T_{y_i} X(w), \beta_i) = \Omega T_{y_{i+1}} X(w)$. By Corollary 4.3 and Equation (6.1), if $\mathcal{A}(\Omega T_{y_i} X(w))$ is free then $\mathcal{A}(\Omega T_{y_{i+1}} X(w))$ is free with the same coexponents. Since $\mathcal{A}(\Omega T_e X(w))$ is free with coexponents $\text{Exp } S(w)$ by Theorem 3.1, $\mathcal{A}(\Omega T_w X(w))$ has the same coexponents.

More generally, if $x \in W(R)$ is a smooth point of $X(w)$, then we can choose a sequence as in the last paragraph with $y_k = x$. It follows from Proposition 3.4 that $-x^{-1}\Omega T_x X(w)$ is coconvex for every smooth point $x \in X(w)^T$, and if $\mathcal{A}(I(w))$ is free then $\mathcal{A}(\Omega T_x X(w))$ is free with the same coexponents.

One difference between geometric and combinatorial Peterson translation is that the weight spaces $\Omega M \subseteq \Omega T_y X$ can be partitioned into positive and negative roots, and the positive roots are a subset of the inversion set $I(y)$.

Proposition 6.3. *If S is a coconvex subset containing an inversion set $I(y)$, and $\alpha \in I(y)$, then $I(r_\alpha y) \subseteq \tau(S, \alpha)$.*

Proof. If $S \subseteq S'$, then $\tau(S, \alpha) \subseteq \tau(S', \alpha)$, so we can assume that $S = I(y)$ and $\alpha \in I(y)$. Suppose $\beta \in I(r_\alpha y)$, and let U be the subspace of V spanned by α and β . Let y_0 be the unique element of $W(R_U)$ with $I(y_0) = I(y)_U$. Note that $r_\alpha \in W(R_U)$, and $\alpha \in I(y_0)$. Since $\beta \in I(r_\alpha y)$, either $r_\alpha \beta \in I(y)$, or $r_\alpha \beta \in R^-$ and $-r_\alpha \beta \notin I(y)$. By construction $r_\alpha \beta \in U$, so if $r_\alpha \beta \in I(y)$ then $r_\alpha \beta \in I(y_0)$ as an element of R_U , and it follows that $\beta \in I(r_\alpha y_0) \subseteq R_U^+$. If $r_\alpha \beta \in R^-$ and $-r_\alpha \beta \notin I(y)$, then $r_\alpha \beta \in R_U^-$, and $-r_\alpha \beta \notin I(y_0)$, so again $\beta \in I(r_\alpha y_0)$.

We can check that the proposition holds for inversion sets in rank 2. Since $\beta \in I(r_\alpha y_0)$,

$$\beta \in \tau(I(y_0), \alpha) = \tau(I(y)_U, \alpha) \subseteq \tau(I(y), \alpha).$$

□

On the geometric side, Proposition 6.3 states that if $S \subseteq \Omega T_y X(w)$ contains $I(y)$ and $\alpha \in I(y)$ then $\tilde{\tau}(S, \alpha)$ contains $I(r_\alpha y)$. This suggests looking at pairs $(S, I(y))$, where S is a coconvex subset of R^+ and $I(y)$ is an inversion set contained in S . The pair $(S, I(y))$ corresponds to the subset $-y^{-1}S$ of $\Omega T_{y^{-1}} X$. Peterson translation by $\alpha \in I(y)$ sends $(S, I(y))$ to $(\tau(S, \alpha), I(r_\alpha y))$.

Example 6.4. *In Example 6.2, we started with the pair $(S, I(w))$, where $S = I(w)$ and $X(w)$ was smooth. We then translated $(S, I(w))$ to the pair $(S(w), \emptyset)$ at the identity. In general we can translate any pair $(S, I(w))$ to some pair (S_e, \emptyset) using only roots in the inversion set. However, S_e is not necessarily a lower order ideal. For example, if we start with the element $w = s_1 s_2 s_3 s_2 s_1$ from Example 3.10 and set $S = I(w)$, then*

$$\tau((I(w), I(w)), \alpha_1 + \alpha_2 + \alpha_3) = (I(w), \emptyset)$$

sends $(I(w), I(w))$ to the identity in one step. It takes at least two steps to translate $I(w)$ to an ideal.

In addition to $I(w)$ and $S(w)$, there are many other ways to construct subsets S of R^+ from geometric constructions on the Schubert variety $X(w)$. The next example gives a family of coconvex sets arising in this way.

Example 6.5. *Let $\Theta_x X(w)$ denote the span of the reduced tangent cone to $X(w)$ at x . If x is a smooth point of $X(w)$, then $\Theta_x X(w) = T_x X(w)$. Also, if $x \in W(R)$ is smooth or G is simply-laced, then $\Theta_x X(w)$ is the space of tangents to T -invariant curves in $X(w)$, and*

$$\Omega \Theta_x X(w) = \{\alpha \in R : x^{-1}\alpha \in R^- \text{ and } r_\alpha x \leq w\}$$

(see [CK03] for more background). If $x \in W(R)$ is a maximal singularity then

$$\Theta_x X(w) = \sum_{\alpha \in R^+ \setminus I(x)} \tilde{\tau}(T_{r_\alpha x} X(w), \alpha)$$

by a theorem of Carrell and Kuttler [CK06, Theorem 1.3]. Since x is a maximal singularity, the points $r_\alpha x$ in this sum are smooth, and consequently $-x^{-1}\tilde{\tau}(\Omega T_{r_\alpha x} X(w), \alpha)$

is coconvex for every $\alpha \in R^+ \setminus I(x)$ by Proposition 3.4 and Example 6.2. Since the intersection of convex sets is convex, the union of coconvex sets is coconvex, and consequently

$$-x^{-1}\Omega_{\Theta_x}X(w) = \bigcup -x^{-1}\tilde{\tau}(\Omega_{T_{r_\alpha x}}X(w), \alpha)$$

is coconvex at every maximal singularity x of $X(w)$. Although the freeness of $\mathcal{A}(\Omega_{\Theta_x}X(w))$ is theoretically resolved by Theorem 2.1, it is an open question to fully characterize the pairs (w, x) with this property.

The author is unaware of any geometric proof of Proposition 3.4, and this might also be an interesting problem.

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E-mail address: `wslofstra@math.ucdavis.edu`